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Gravitational scattering in the ADD model revisited

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Abstract. It is argued that the assumption that the standard model particles live on a finite brane in the ADD model does in itself imply a finite propagator for virtual Kaluza–Klein mode exchange. The part of the propagator relevant for large distance scattering is cut-off-independent for scattering at distances large compared to the brane width. The matrix element corresponding to this part can also, at least for an odd number of extra dimensions, be Fourier transformed to position space, giving back the extra-dimensional version of Newton's law. For an even number of extra dimensions a corresponding result is found by requiring that Newton's law should be recovered.

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1 Introduction

The ADD model [1–3] aims at explaining the hierarchy between the weak scale and the Planck scale. This is done by introducing extra, compactified, dimensions in which only gravity is allowed to propagate. At distances small compared to the compactification radius, but large compared to the Planck length and the thickness of the brane, the gravitational force will be enhanced and behave essentially as in a $4+n$ dimensional world, where n is the number of extra dimensions. If the extra dimensions are large enough, the enhanced gravitational force opens up the possibility of gravitational scattering and black hole production at present, or soon upcoming, collider experiments.

To quantify the amount of gravitational interaction, the theory was put on a perturbative field-theoretical basis in [4, 5]. The perturbations of the extra-dimensional part of the metric enter as massive Kaluza–Klein (KK) modes in the Lagrangian. When these modes are internal states they have to be summed over, which leads the authors of [4, 5] to the following divergent propagator integral:

$$\sum_{\bar{l}} \frac{1}{-m_{\bar{l}}^2+k^2} \sim R^n \int \frac{m^{n-1}}{-m^2+k^2}\,\mathrm{d}m\,. \qquad (1)$$

Here $\bar{l}$ enumerates the allowed momenta, $m_{\bar{l}}$, in the extra dimensions, m is the absolute value of $m_{\bar{l}}$, R is the compactification radius[1] and k^2 is the momentum squared of the $3+1$-dimensional part of the propagator. (This object, without Lorentz structure, will somewhat sloppily be referred to as a propagator.) For the above approximation to be valid, the compactification radius of the extra dimensions clearly has to be large compared to other relevant length scales. As it stands, the integral (1) is explicitly divergent for $n \geq 2$. However, when arriving at (1), the physical condition that the standard model fields are confined to the brane was not taken into account.

Instead the problem of the divergent integral was approached in [4, 5] by introducing a cut-off M_s, argued to be of the same order of magnitude as the fundamental Planck scale, M_p. (A physical motivation for a cut-off was later considered in [6, 7] by the introduction of a brane tension, and various mathematical shapes of cut-offs have been discussed in [8].) For $n>2$, and exchanged momentum small compared to M_s, the Kaluza–Klein summation of t-channel (or s-channel) amplitudes then gave a propagator behaving as

$$\frac{1}{n-2} R^n M_\mathrm{s}^{n-2} \sim \frac{1}{G_{\mathrm{N}(4)}} \frac{1}{n-2} \frac{M_\mathrm{s}^{n-2}}{M_\mathrm{p}^{n+2}}\,, \qquad (2)$$

where $G_{\mathrm{N}(4)}$ is the ordinary $3+1$-dimensional Newton's constant related to R and M_p via $G_{\mathrm{N}(4)}^{-1} \approx R^n M_\mathrm{p}^{n+2}$. (In [4] M_p and M_s are taken to be the same in the calculation of the propagator.) In the Born approximation the cross section would then be given by [9]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \sim \frac{s^3}{(n-2)^2} \left(\frac{M_\mathrm{s}^{n-2}}{M_\mathrm{p}^{n+2}}\right)^2 F(\mathrm{spin}, z)\,, \qquad (3)$$

where z is the cosine of the scattering angle in the center of mass system, s the squared sum of the incoming particles momenta and F a function taking the spin dependence into account.

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[1] R is here used to denote the compactification radius, rather than the compactification circumference.

A different approach to calculating the influence of the Kaluza–Klein modes was presented in [2] when deriving Newton's law. In this case the summation of Kaluza–Klein modes was performed in the classical limit, *after* Fourier transforming our normal momentum space to coordinate space [2], giving rise to the expected $3+n$ dimensional version of Newton's law,

$$\frac{V(r)}{m_1 m_2} \sim \int_{-\infty}^{\infty} \mathrm{d}\overline{m} \int_{-\infty}^{\infty} \mathrm{d}\overline{k} \frac{1}{\overline{m}^2+\overline{k}^2} \mathrm{e}^{\mathrm{i}\overline{k}\cdot\overline{r}} \sim \int_0^{\infty} \mathrm{d}m m^{n-1} \frac{\mathrm{e}^{-mr}}{r} \sim \frac{1}{r^{n+1}}, \tag{4}$$

where $r=|\overline{r}|$. Equations (2) and (4) may seem to contradict each other. Especially, (2) gives the same form of the gravitational scattering potential, regardless of the number of extra dimensions, namely a δ-function at $\overline{r}=\overline{0}$. Equation (4), on the other hand, gives a different scattering behavior for a different number of extra dimensions. We will see in the next section that k-dependent correction terms to (2) are important for the classical limit as the $1/r^{n+1}$-potential cannot be recovered by Fourier transforming the non-relativistic matrix element corresponding to (2) to ordinary position space.

We will later implement the physical condition that standard model fields live on a brane via a Fourier transform. To be consistent with this we should really add a factor $\mathrm{i}\overline{y}\cdot\overline{m}$, where $\overline{y}$ is the coordinate in the extra dimensions, in the exponent and then integrate over a narrow distribution in $\overline{y}$. However, as the $\overline{m}$ integral is explicitly convergent when evaluated after the $\overline{k}$ integral as in (4), the addition of $\mathrm{i}\overline{y}\cdot\overline{m}$ in the exponent would not change the result in the classical limit when the distance is much larger than the brane thickness. In essence, what will be done in this paper is nothing but calculating (4) in the reversed order, starting with the m integral.

The Fourier transform to position space with respect to the extra dimensions is performed (for an odd number of extra dimensions) in Sect. 2. In Sect. 3 we show that the parts of the propagator relevant for the large distance limit can be further Fourier transformed to position space with respect to our normal dimensions, giving back the extra-dimensional version of Newton's law. We also find a corresponding "KK-summed propagator" for an even number of extra dimensions by requiring that Newton's law should be recovered. Finally we summarize and conclude in Sect. 4.

2 Fourier transformation to position space in the extra dimensions

As the standard model fields are assumed to live on the brane, any measurement of gravitational scattering will be in position space with respect to the extra-dimensional coordinates. We therefore search for the corresponding propagator. To find it we introduce a coordinate $\overline{y}$, with absolute value y, in the extra dimensions. Later we will be interested in a narrow distribution around $\overline{y}=\overline{0}$ corresponding to a small extension of the standard model fields into the bulk. Searching for a propagator that is in position space with respect to the extra dimensions thus enables us to impose the condition of locality. As the standard model fields live on a brane they will only be sensitive to KK modes that overlap with the brane.

The matrix element of a *single* KK mode exchange (in momentum space with respect to the ordinary and extra dimensions) is $\propto 1/(-m^2+k^2)$ in the non-relativistic limit. In position space, if the extra dimensions were not compactified, this would lead to a propagator proportional to

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int \frac{\mathrm{d}^n \overline{m}}{(2\pi)^n} \frac{1}{-m^2+k^2} \mathrm{e}^{-\mathrm{i}k\cdot r} \mathrm{e}^{\mathrm{i}\overline{m}\cdot\overline{y}}. \tag{5}$$

Since the extra dimensions are compactified, a complete set of states is given by the discrete set of standing waves enumerated by $\overline{l}$, and we instead have

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{\overline{l}} \frac{1}{-m_{\overline{l}}^2+k^2} \mathrm{e}^{\mathrm{i}k\cdot r} \mathrm{e}^{\mathrm{i}\overline{l}\cdot\overline{y}/R}. \tag{6}$$

For collider phenomenology we are, however, interested in the case when we are in momentum space with respect to our ordinary dimensions, leaving us with

$$\begin{aligned} D(k,y) &= \sum_{\overline{l}} \frac{1}{-m_{\overline{l}}^2+k^2} \mathrm{e}^{\mathrm{i}\overline{l}\cdot\overline{y}/R} \\ &\approx R^n S_{n-1} \int_0^{\pi} \sin(\theta)^{n-2}\, \mathrm{d}\theta \\ &\quad \times \int_0^{\infty} \mathrm{d}m \frac{m^{n-1}}{-m^2+k^2} \mathrm{e}^{\mathrm{i}my\cos(\theta)} \\ &= R^n S_{n-1} \int_{-1}^{1} \mathrm{d}\cos(\theta)(1-\cos^2(\theta))^{(n-3)/2} \\ &\quad \times \int_0^{\infty} \mathrm{d}m \frac{m^{n-1}\mathrm{e}^{\mathrm{i}my\cos(\theta)}}{-m^2+k^2}, \end{aligned} \tag{7}$$

for $n \geq 3$. Here S_{n-1} is the surface of a unit sphere in $n-1$ dimensions (from integration over the angles on which the integrand does not depend), and the factor R^n comes from the density of Kaluza–Klein modes. This can also be seen by taking the Fourier transform instead of the sum in (1). For an odd number of extra dimensions (7) can, with $x=\cos(\theta)$, be rewritten as

$$\begin{aligned} D(k,y) = R^n S_{n-1} \int_{-1}^{1} \mathrm{d}x (1-x^2)^{(n-3)/2} \int_0^{\infty} \mathrm{d}m \\ \times \left(-\sum_{j=0}^{(n-3)/2} m^{n-3-2j} k^{2j} + \frac{(k^2)^{\frac{n-1}{2}}}{-m^2+k^2} \right) \mathrm{e}^{\mathrm{i}myx}. \end{aligned} \tag{8}$$

From this form we see that the terms in the sum are either k-independent, and therefore are Fourier transformed to δ-functions at $\overline{r}=\overline{0}$ in ordinary position space, or are *even* powers of k and are Fourier transformed to derivatives of the δ-function. (This is easily seen by performing the Fourier transform from k-space to r-space component by component and recalling that the Fourier transform from k_x to r_x of k_x^{2j} is $\propto \delta^{(2j)}(r_x)$.)

If we explicitly consider the brane to have a finite narrow distribution (such as a Gaussian) around $\overline{y} = \overline{0}$ in the extra dimensions, the $1/r^{n+1}$ law will be modified at distances similar to the brane thickness. In particular, if the distance is short enough for the distribution to be approximated by a constant $\left(r \ll \sigma \text{ for } \mathrm{e}^{-y^2/(2\sigma^2)}\right)$, the dominating correction will be $\propto 1/r$. We note that these corrections come from the *last* term in the parentheses in (8).

In k-space the terms in the sum will for a finite brane give (possibly) large but finite contributions. These terms, which depend on the extension of the brane, are important when the wave functions overlap and will then give an interaction similar to that from (3) (until the unitarity condition sets in), but they will not be further investigated here. Instead we concentrate on the term relevant for large distance and large energy scattering (the classical limit) coming from the last term in the parentheses in (8). The contribution to the integral from this term, here called $\bar{D}(k)$, is easily evaluated in the limit of small y (corresponding to a narrow distribution) and is given by

$$\bar{D}(k) \approx R^n \frac{\pi S_n}{2} \left(\sqrt{-k^2}\right)^{n-2} (-1)^{\frac{n+1}{2}} . \qquad (9)$$

It is easy to show that this result holds also for one extra dimension, and therefore for any odd number of extra dimensions. (Note that this is a positive *odd* power of $|k|$, i.e. it is non-analytic in e.g. k_x. This is essential, since if it was even it would have as its Fourier transform a derivative of the δ-function.)

For an even number of extra dimensions the propagator in (7) is also turned finite by (for example) a Gaussian distribution in y-space corresponding to a Gaussian distribution in m-space. In this case there is however no finite term which directly corresponds to the classical potential.

The expression (9), which is in momentum space with respect to our normal dimensions, and in position space with respect to the extra dimensions, has the following properties.

1. It gives back Newton's law, (4). This will be demonstrated in Sect. 3.
2. It depends on the number of extra dimensions in a non-trivial way, such that, as the gravitational force increases faster with smaller distance in position space for many extra dimensions, this is reflected in a faster increase with larger k in momentum space.
3. It does not depend on an arbitrary cut-off as long as the cut-off ($\approx 1/$(brane thickness)) is much larger than k. This implies that it is not dominated by metric perturbations of the scale $1/M_\mathrm{p}$ for scatterings corresponding to much larger distances. This is the case for (1) integrated to $M_\mathrm{s} \approx M_\mathrm{p}$.
4. It is the part of the propagator which is argued to contribute to the all order exponentiated eikonalized amplitude in [10] (apart from what appears to be a sign misprint).

3 Fourier transformation to position space in our ordinary dimensions

To obtain the $3+1+n$ dimensional version of Newton's law we take the classical limit such that the energy is given by the mass, multiply with the coupling constant $4\pi G_{\mathrm{N}(4)}$ and Fourier transform (9) to position space. Using $\kappa = |\overline{k}|$ we have

$$\begin{aligned} \frac{V(r)}{m_1 m_2} &= 4\pi G_{\mathrm{N}(4)} R^n \frac{\pi S_n}{2} \mathrm{i}^{n+1} \frac{S_3}{2} \frac{1}{(2\pi)^3} \\ &\quad \times \int_{-1}^{1} \mathrm{d}\cos(\theta) \int_0^\infty \mathrm{d}\kappa \kappa^2 \kappa^{n-2} \mathrm{e}^{\mathrm{i}\kappa r \cos(\theta)} \\ &= 4\pi G_{\mathrm{N}(4)} R^n \frac{\pi S_n S_3}{4(2\pi)^3} \mathrm{i}^{n+1} \frac{1}{\mathrm{i}r} \\ &\quad \times \int_0^\infty \mathrm{d}\kappa \kappa^{n-1} (\mathrm{e}^{\mathrm{i}\kappa r} - \mathrm{e}^{-\mathrm{i}\kappa r}) . \end{aligned} \qquad (10)$$

This can be evaluated by the introduction of a small convergence factor,

$$\begin{aligned} \frac{V(r)}{m_1 m_2} &= 4\pi G_{\mathrm{N}(4)} R^n \frac{\pi S_n S_3}{4(2\pi)^3} \mathrm{i}^{n+1} \frac{1}{\mathrm{i}r} \\ &\quad \times \lim_{\epsilon \to 0} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1} \int_0^\infty \mathrm{d}\kappa \left[\frac{\mathrm{e}^{\mathrm{i}\kappa r - \epsilon\kappa}}{\mathrm{i}^{n-1}} - \frac{\mathrm{e}^{-\mathrm{i}\kappa r - \epsilon\kappa}}{(-\mathrm{i})^{n-1}}\right] \\ &= G_{\mathrm{N}(4)} R^n \frac{\pi S_n (4\pi)^2}{4(2\pi)^3} \mathrm{i}^{n+1} \frac{1}{\mathrm{i}r} \frac{1}{\mathrm{i}^{n-1}} \\ &\quad \times \lim_{\epsilon \to 0} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1} (-1) \left[\frac{-\mathrm{i}}{r + \mathrm{i}\epsilon} - \frac{\mathrm{i}(-1)^{n-1}}{r - \mathrm{i}\epsilon}\right] \\ &= -G_{\mathrm{N}(4)} R^n S_n \frac{\Gamma(n)}{r^{n+1}} , \end{aligned} \qquad (11)$$

where the last step only is valid for an odd number of extra dimensions. This is the same result as in [2] (apart from a minus sign which is neglected in [2]; the gravitational potential must be attractive), i.e. gravitational scattering enhanced by the large density of Kaluza–Klein modes, corresponding to a large coupling constant.

The strategy so far has been to start from the propagator and to argue that we can get back Newton's law. Clearly this argument could be turned upside down. Using the result in (4) [2], we could alternatively search for the propagator giving the expected potential when Fourier transformed to position space. Again this would give us a term of the form $|k|^{n-2}$ for an odd number of extra dimensions. For an even number of extra dimensions we settle with this argument, showing that the term

$$\bar{D}(k) \approx R^n \frac{S_n}{2} (-1)^{\frac{n-2}{2}} \left(\sqrt{-k^2}\right)^{n-2} \ln(-k^2) \qquad (12)$$

gives the desired form. The logarithm of a dimension-full quantity may seem disturbing. Replacing it by $\ln(-k^2/k_0^2)$ for some k_0 we note that the $k^{n-2}\ln(k_0^2)$ term would only contribute at $\overline{r} = \overline{0}$ when Fourier transformed to position space. This means that, just as in the case of an odd number of extra dimensions, we have a local interaction which depends on properties of the brane. We also note that, just as for odd n, requiring the standard model particles to live on a thin brane by introducing a (narrow) distribution in y-space we would have a (wide) distribution in m-space giving a (large) finite value for the propagator.

Fourier transforming the non-relativistic version of (12) to position space we find the potential

$$\begin{aligned}\frac{V(r)}{m_1 m_2} &= 4\pi G_{\mathrm{N}(4)} R^n \frac{S_n}{2} (-1)^{\frac{n-2}{2}} \frac{S_3}{2} \frac{1}{(2\pi)^3} \\ &\quad \times \int_{-1}^{1} \mathrm{d}\cos(\theta) \int_0^\infty \mathrm{d}\kappa \kappa^2 \kappa^{n-2} \ln(\kappa^2) \mathrm{e}^{\mathrm{i}\kappa r \cos(\theta)} \\ &= G_{\mathrm{N}(4)} R^n \frac{S_n (4\pi)^2}{4(2\pi)^3} (-1)^{\frac{n-2}{2}} \frac{1}{\mathrm{i}r} \\ &\quad \times \int_0^\infty \mathrm{d}\kappa \kappa^{n-1} \ln(\kappa^2) 2\mathrm{i} \sin(\kappa r) \,. \end{aligned} \tag{13}$$

For n even we get after evaluating the integral [11]

$$\frac{V(r)}{m_1 m_2} = -G_{\mathrm{N}(4)} R^n S_n \frac{\Gamma(n)}{r^{n+1}} \,. \tag{14}$$

This is the multidimensional version of Newton's law. We also note that (12) is the part of the propagator argued to contribute to the all order eikonalized amplitude in [10].

4 Conclusion and outlook

We have shown that requiring the standard model particles to live on a finite brane leads to a convergent result for the KK propagator. The part of the propagator which is seen to be relevant for large energy and large distance scattering can also (at least for an odd number of extra dimensions) be Fourier transformed to position space with respect to our ordinary coordinates, giving back Newton's law. For an even number of extra dimensions we have found a similar expression by requiring that we should get back Newton's law in the classical limit, when all coordinates are Fourier transformed to position space.

It should be pointed out that, although the part of the tree-level amplitude relevant for large distance scattering is calculated to a finite value, we cannot use (9) and (12) for calculating cross sections with the Born approximation. This is so because at low energies the thickness of the brane is important, and at high energies an all order eikonal summation has to be performed [10]. This calculation is, however, more reliably performed by keeping a finite brane width throughout the calculations and will be considered separately.

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References

1. N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B **429**, 263 (1998) [hep-ph/9803315]
2. N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Rev. D **59**, 086 004 (1999) [hep-ph/9807344]
3. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B **436**, 257 (1998) [hep-ph/9804398]
4. T. Han, J.D. Lykken, R.-J. Zhang, Phys. Rev. D **59**, 105 006 (1999) [hep-ph/9811350]
5. G.F. Giudice, R. Rattazzi, J.D. Wells, Nucl. Phys. B **544**, 3 (1999) [hep-ph/9811291]
6. M. Bando, T. Kugo, T. Noguchi, K. Yoshioka, Phys. Rev. Lett. **83**, 3601 (1999) [hep-ph/9906549]
7. T. Kugo, K. Yoshioka, Nucl. Phys. B **594**, 301 (2001) [hep-ph/9912496]
8. G.F. Giudice, A. Strumia, Nucl. Phys. B **663**, 377 (2003) [hep-ph/0301232]
9. D. Atwood, S. Bar-Shalom, A. Soni, Phys. Rev. D **62**, 056 008 (2000) [hep-ph/9911231]
10. G.F. Giudice, R. Rattazzi, J.D. Wells, Nucl. Phys. B **630**, 293 (2002) [hep-ph/0112161]
11. I.M. Gradshteyn, I. Ryshik, Table of Integrals, Series, and Products (Academic Press, USA, 1994)